THE EXISTENCE RESULTS FOR SOLUTIONS OF INDEFINITE SCALAR CURVATURE PROBLEM

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ABSTRACT. In this paper, we consider the indefinite scalar curvature problem on R^n. We propose new conditions on the prescribing scalar curvature function such that the scalar curvature problem on R^n (similarly, on S^n) has at least one solution. The key observation in our proof is that we use the bifurcation method to get a large solution and then after establishing the Harnack inequality for solutions near the critical points of the prescribed scalar curvature and taking limit, we find the nontrivial positive solution to the indefinite scalar curvature problem.

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1. INTRODUCTION

In this work, we consider the indefinite scalar curvature problem both on S^n and on R^n ($n \geq 3$). One may know that there are very few existence results on such a problem (see [41] for a use of the variation method). Since the scalar curvature problem can be reduced to that of R^n, we mainly consider the problem on R^n. The problem then is equivalent to solving the following equation

$$(1) \qquad -\Delta u = K(x)u^p, \;\; u > 0 \;\; in \;\; R^n,$$

where $p = \frac{n+2}{n-2}$, $n \geq 3$, and K is a sign-changing smooth function on R^n, which has *isolated critical points* and with

$$(2) \qquad \lim_{|x|\to\infty} K(x) = K_\infty < 0.$$

Other extra conditions on both the zero set of K and positive part of K, which shall be specified below. Roughly speaking, there three kinds of methods are used in the study of the positive or negative scalar curvature problems. One is the best constant method of Th.Aubin's school in the search of Sobolev inequalities on manifolds (see [3] and [4]). The second one in attacking this problem is the use of the critical point theory (see [6] and [7]). The third one has been proposed by R.Schoen who prefers to use the degree counting method starting from a subcritical problem and this method has

been improved by Chang-Yang, Chen-Li, Chen-Lin, and Y.Y.Li. Unlike the previous studies of these problems, where the authors only have considered the existence problem when K is a positive/negative smooth function or an apriori bound for solutions when K is sign-changing (see the famous works, for example, [7],[15], [17],[19], [32], [49]), we use the bifurcation method to attack this problem and we obtain a new result. Related existence result about critical indefinite elliptic problems on bounded domains is considered in [26], where a different assumption is used. For subcritical indefinite elliptic problems, this bifurcation point of view was taken in the previous works of L.Nirenberg, al et [8], and first named author [22], just named a few here. As it is well-known, the key step in the application of bifurcation theory developed by P.Rabinowitz [46] and Crandall-Rabinowitz [21] is the apriori estimate for solutions. For this purpose, our analysis in [24] will play a role in the study of indefinite scalar curvature problem in R^n.

We now introduce our problems. Let $\mu \in [0, L]$. We study the following equation:

$$(3) \qquad\qquad -\Delta u = \mu u + K(x)u^p, \ \ u > 0 \ \ in \ \ R^n,$$

When K is positive and away from the critical points of K, we have the uniform bound thanks to the works of Caffarelli-Gidas-Spruck, Chang-Yang, Chen-Li, Chen-Lin, Y.Y.Li, and Schoen in 90's.

Assume that $0 \in B_{3R}$ is the isolated critical point of K. To find apriori estimate of positive solutions to scalar curvature problem, the basic assumption for K is the $(\beta - 1)$-flatness condition as introduced by Y.Y.Li and Chen-Lin, where the condition says that

(i) (*n-2-Flatness*). Assume $K \in C^{n-2}(B_R(0)$. For any $\epsilon > 0$, there exists a neighborhood B_r of 0 such that

$$|\nabla^s K(q)| \le \epsilon |\nabla K(q)|^{\frac{l-s}{l-1}}, \quad for \quad q \in B_r,$$

where $l = n - 2$.

If K has $n - 2$-flatness condition at its critical points, then the solutions to the scalar curvature problem (1) enjoy the apriori bound on the positive part of K (see the proof of Theorem 1.9 in [38]). It is also easy to see that the similar result is also true for equation (3).

We make two kinds of assumptions in this article. One is the $n-2$-flatness for K when $\Delta K(0) \le 0$. As we pointed out above, in this case, we have the apriori bound for solutions (in particular we have the Harnack inequality, 17 below).

The other one is for $\Delta K(0) \ge 0$. The assumption on K in this case is the following two statements (K):

(ii). Let $= \min\{n - 2, 4\}$, $|K(x)_{C^\beta(B_{3R})} \le C_1$ and $1/C_1 \le K(x) \le C_1$ in B_{3R} for some $C_1 > 0$;

and

(iii). There exist $s, D > 0$ such that for any critical point $x \in B_{2R}$ of K, $\Delta K \ge 0$ in $B(x, sR)$ for n = 4, 5, 6 and $\Delta K \ge D$ in $B(x, sR)$ for $n \ge 7$.

We shall mainly consider this case where these conditions are assumed on K. Assuming the possible blow up of solutions near the critical points, we can set up a Harnack inequality (see Theorem 8 below).

As it is well-known that one of the key part of the scalar curvature problem is the Harnack estimate for positive solutions to (3) on the positive part of the function K. Roughly speaking, one of main part of our result is

Main Result 1. *Assume that $\mu \geq 0$. Assume (ii) and (iii). We derive a Harnack inequality for solutions to (3) on B_{3R}. If the positive scalar curvature function $K(x)$ is sub-harmonic in a neighborhood of each critical point and the maximum of u over B_R is comparable to its maximum over B_{3R}, then the Harnack type inequality can be obtained. Furthermore, assume $\mu > 0$, then, as a consequence of Harnack estimate, we can have an uniform bound for positive solutions to (3) on B_{3R}.*

The precise statement of Harnack inequality is stated in next section. We point out that for the Harnack inequality to be true, we only need a weaker assumption than $n - 2$-flatness on K, however, we shall not formulate it but refer to [32].

To obtain the an apriori bound near the zero set of K, we need some notations.

$$D_- = \{x \in R^n; K(x) < 0\}, \quad D_+ = \{x \in R^n; K(x) > 0\}, \quad K^+ = \max\{K(x), 0\}.$$

We now give the assumption

(iv). We assume that $D_+ = \{x \in R^n; K(x) > 0\}$ is a *bounded domain* with smooth boundary and there exist two positive constants c_1, c_2 such that near $D_0 = \{x \in R^n; K(x) = 0\}$, we have

$$0 < c_1 \leq K(x) \leq c_2.$$

Using this assumption and the moving plan method, Lin (see corollary 1.4 in [38]) proved that there is an apriori bound for solutions to (1). With an easy modification based on some argument in the work of Chen-Lin [19], we can directly extend Lin's result to our case (3). Since the proof is straightforward, we omit the detail here. We just remark that there are other conditions which make the apriori bound hold true for our equation (3). For these, one may see the remarkable works of Chen-Li [17] and [38]. In [23], another condition was proposed that if further assume that there is a continuous function $k : \bar{D}_+ \to (0, \infty)$, which is bounded away from zero near the boundary ∂D_+ and a constant

$$\gamma > p + 1$$

such that near a neighborhood of ∂D_+ in D_+, we have

(4) $$K^+(x) = k(x)[dist(x, \partial D_+)]^\gamma + h.o.t,$$

then by using the moving plane method, the first named author and S.J.Li [23] showed that there is a uniform L^∞ bound for solutions to (3) near the boundary of D_+ in R^n. We remark that there are also some other conditions

which make the uniform bound to be true ([17] and [40]). The proof of Main Result 1 with $n - 2$-flatness is similar to the proof of Theorem 1.9 in [38]. So for this part, we omit the proof.

To obtain the uniform bound on the negative part of the function K, we can use the boundary blow-up solution to get the bound. By now it is a standard method, one may see [24].

Using all these uniform bound results, we then obtain the an uniform apriori estimate for positive solutions to (3).

Theorem 2. *(I). Let $\mu > 0$. Assume that the conditions (i-1v) are true for K. Then there is an uniform constant $C > 0$ such that for any positive solution u to (3),*

$$|u|_{L^\infty(R^n)} \le C.$$

(II). Assume (i) is true at every critical point of K on its positive part, and assume (iv) and (4). Then we have an uniform bound for positive solutions to (1).

The proof of part (II) of Theorem 2 is similar to the proof of Theorem 1.9 in [38]. So we omit the proof.

Actually according to the blow up analysis due to Schoen (see also [32]), we only need to treat the uniform bound near the critical point of K on the positive part of K. This will be studied by proving the Harnack inequality (see Theorem 1). Using the standard elliptic theory we further know that for any solution u,

$$|u|_{C_b^2(R^n)} \le C,$$

Using the bifurcation theory developed in [22] (see also [42] for Yamabe problem with Dirichlet condition), we have the main result of this paper.

Theorem 3. *Assume the conditions as in Theorem 2. There is at least one positive solution to the scalar curvature problem (1).*

To prove Theorem 3, we need to study the following problem in the large ball $B_R(0)$ for $\mu > 0$

$$(5) \qquad\qquad -\Delta u = \mu u + K(x)u^p, \ \ u > 0 \ \ in \ \ B_R(0),$$

with the Dirichlet boundary condition

$$(6) \qquad\qquad\qquad u = 0, \ \ on \ \ \partial B_R(0).$$

We can show

Theorem 4. *Assume the conditions as in Theorem 2. There are two constants $\Gamma_1 < \Gamma$ such that for each $\mu \in (\Gamma_1, \Gamma)$, there is at least two positive solutions to the problem 8 with the Dirichlet boundary condition. Furthermore, the solutions are uniformly bounded with the bound independent of $R >> 1$.*

As a sharp different, we show that

Theorem 5. *Assume the conditions (i-iv) for K are true in R^n. There exists a positive constant τ^* such that for each $\tau \in (0, \tau^*)$, (3) in R^n has a minimal positive solution u_τ and at least another positive solution u^τ, which is not in the order interval $[0, u_\tau]$, and there is no positive solution when $\tau > \tau^*$. Moreover, $u_{\tau_1} \leq u_{\tau_2}$ for $0 < \tau_1 < \tau_2 < \tau^*$ and there is a uniform constant C depending only on τ and K such that for each $\tau \in (0, \tau^*)$ and for any positive solution u to (7) in R^n, we have*

$$|u(x)| \leq C, \;\; for \;\; x \in R^n.$$

Furthermore, the bound is independent of $\tau \geq 0$ if x is outside of a large ball.

One may see [43] for the existence result of the scalar curvature problem with nontrivial Dirichlet boundary condition.

Here is the plan of the paper. In the first part of this paper, we prove the existence results based on the apriori estimate. Then we prove the apriori estimate on the non-positive part of the scalar curvature K in section 3. In section 4, we obtain apriori estimate based on Harnack inequality on the positive part of the scalar curvature K. The main apriori estimate is the Harnack inequality (Theorem 1), whose proof is contained in the remaining sections. In the proof of Theorem 1, we shall use the some arguments from Chen-Lin [20] and Li-Zhang [34] (see also [53]).

2. Proofs of Theorem 3 and others assuming the apriori bound

In this section, we give the proof of Theorem 3 according to the bifurcation method used in [22] (see also [8],[10] and [25] for related problems). In particular, we recall the following result in [8].

Theorem 6. *For any $p > 1$ and any continuous function m on the closure of the bounded smooth domain Ω of R^n, if ϕ denotes the eigenfunction associated with μ_1, which is the principal eigenvalue of the operator*

$$-\Delta + m(x)$$

on Ω with the zero Dirichlet boundary condition, and $K(x)$ takes both positive and negative values, the following assertion holds. If i) $\int_\Omega K(x)\phi^{p+1}(x)dx < 0$, then there exists $\tau^* = \tau^*(\Omega) > \mu_1$ such that problem

$$(7) \qquad\qquad -\Delta u + (m(x) - \tau)u = K(x)u^p, \;\; in \;\; \Omega,$$

with the zero Dirichlet boundary condition or zero Newmann boundary condition, has a solution for every $\tau \in [\mu_1, \tau^*)$, while no solution exists for $\tau > \tau^*$. Conversely, condition i) is also a necessary condition for existence of solutions.

We remark that the existence part of the proof is obtained by a constrained minimization method and the necessary part is derived from a generalized Picone identity. For our case where $m(x) = 0$, $\Omega = B_R(0)$ for large $R > 1$, and $p = \frac{n+2}{n-2}$, the condition (i) can be easily verified.

In fact, let $\phi_1(x)$ be the first eigenvalue of $-\Delta$ on the unit ball $B_1(0)$, we have $\phi_R(x) = \phi_1(x/R)$ and $\mu_R = \mu_1/R^2$ on B_R. Then

$$\int_{B_R} K(x)\phi_R^{p+1}(x)dx = R^n \int_{B_1(0)} K(Ry)\phi_1(y)^{p+1}dy$$

$$= R^n(\int_{B_{R_0/R}} + \int_{B_1-B_{R_0/R}})K(Ry)\phi_1^{p+1}.$$

Note that $K(Ry) \le -A < 0$ for $y \in B_1 - B_{R_0/R}$ and $|K(Ry)| \le B$ in B_1, hence we have

$$\int_{B_1-B_{R_0/R}} K(x)\phi_1^{p+1} \le -A \int_{B_1-B_{R_0/R}} \phi_1^{p+1} \to -A \int_{B_1} \phi_1^{p+1} < 0$$

and

$$\int_{B_{R_0/R}} K(Ry)\phi_1^{p+1} \to 0, \ R \to \infty,$$

Using this we have

$$\int_{B_R} K(x)\phi_R^{p+1}(x)dx < 0$$

for large $R > 0$.

Actually we can extend their result when $p = \frac{n+2}{n-2}$ and when the large ball B_R is contained in Ω in the following way.

Theorem 7. *Assume the conditions (i-iv) for K are true in Ω. For each $\tau \in (\mu_1, \tau^*)$, (i1) (7) has a minimal positive solution $u_{\tau,\Omega}$ in the sense that any positive solution u to (7) satisfies $u \ge u_{\tau,\Omega}$ in Ω; (i2) there is a uniform constant depending only on R and K such that*

$$|u(x)| \le C, \ for \ x \in \Omega$$

for any positive solution to (7) in Ω; (i3) moreover when $\Omega = B_R$ and setting $\tau_R^* = \tau^*(B_R)$ and $u_{\tau,R} = u_{\tau,B_R}$, we have

$$\tau_R^* \le \tau_r^*, \ r < R$$

and

$$u_{\tau_1,r} \le u_{\tau_2,R}, \ (\tau_1, r) < (\tau_2, R)$$

Proof. (i1) and (i3) can be proved in the same way as in [22]. Assume that u is any positive solution to (7) in Ω. To prove (i2), we note that by the results obtained in [24], [23], and [32], u is uniformly bounded away from the critical points of K in the positive part of K. However, near the critical point of K in positive part of K, we have the Harnack inequality (see the coming sections). Since u has a uniform lower bound given by $\epsilon\phi$ on Ω, we have the uniform bound for u.

$\square$

Using Theorem 7, we can prove Theorem 5. We remark that in the case of Theorem 5, all the positive solutions have the behavior at infinity:

$$\lim_{|x|\to\infty} u(x) = -K_\infty/\mu,$$

and with this and our Harnack inequality 17 (see below), we know that all solutions to (3) are uniformly bounded. The proof of this result is omitted here since the proof of the existence part is in [22].

We are now in the position to prove Theorem 3.

Proof. Take a fixed $0 < \tau_1 < \tau^*$ and sequence $0 < \tau_j \to 0$ with the solution sequence $u_j = u^{\tau_j}$. We then define

$$u(x) = \lim_{j\to\infty} u_j(x).$$

We want to show that u is a non-trivial smooth positive solution to (eq1) with $\mu = 0$. According to our choice of u_j, we have a bounded sequence of point z_j such that

$$u_j(z_j) > u_{\tau_1}(z_j).$$

Assume that $\lim_{j\to\infty} z_j = z^*$. we can assume that

$$u_j(z_j) > \frac{1}{2} u_{\tau_1}(z^*).$$

Using the standard Harnack inequality we have a uniform ball B_r such that

$$\min_{B_r} u_j \geq \frac{1}{200} u_{\tau_1}(z^*).$$

Then using the Harnack inequality in Theorem 8 in section 5 again, we have u_j is uniformly bounded in the whole space. Using the elliptic theory, we have a convergent subsequence, still denoted by u_j. Hence the limit u is a positive bounded solution to (3) on R^n with $\mu = 0$. This completes the proof of Theorem 3. $\square$

3. APRIORI BOUND ON NON-POSITIVE PART

Fix $\delta > 0$. On the domain $\{x \in R^n, K(x) \leq -\delta\}$, one can see easily that for small $R > 0$, the boundary blow up function

$$B(x) = \frac{1}{(R^2 - |x - x_0|^2)^a}, \quad x \in B_R(x_0)$$

is a super-solution to (3) in the ball $B_R(x_0)$. Hence, we obtain the uniform bound for all solutions on this part.

On the domain $\{x \in R^n, |K(x)| \leq \delta\}$, one can use the moving plane method and blow up trick to get the uniform bound for all solutions on this part too. For more detail, one may refer [18]or [38] to the proof.

4. APRIORI BOUND ON POSITIVE PART BASED ON HARNACK INEQUALITY AND POHOZAEV IDENTITY

In this section, we firstly recall the general blow up trick often used in the study of scalar curvature problem on bounded smooth domains. Then we show the uniform bound for solutions to (3).

For fix $y \in R^n$, we define

$$v(x) = v_y(x) = \tau^{\frac{n-2}{2}} u(\tau x + y),$$

which satisfies that

$$-\Delta v = \lambda \tau^2 u + K(\tau x + y) v^p.$$

Fix $R > 0$. Consider $f(x) = (R - |x|) u^{1/a}$. Assume not. Then there are a sequence of solutions $\{u_j\}$ and a sequence $\{x_j\}$ ($|x_j| \leq 1$) such that

$$f_j(x_j) = (R - |x_j|) u_j(x_j)^{1/a} = \max(R - |x|) u_j^{1/a} > j \to \infty,$$

which implies that

$$u_j(x_j) \to \infty.$$

Let $\lambda_j = u_j(x_j)^{-1/a}$. Let

$$v_j(x) = u_j(x_j)^{-1} u_j(x_j + \lambda_j x).$$

Note that for $|x| \leq R/2$, we have

$$u(x) \leq 2^{2a} u_j(x_j).$$

The corresponding domain for $|v_j(z)| \leq 2^{2a}$ contains the ball

$$\{z; |z| \leq f_j(x_j)\} \to R^n.$$

Using the standard elliptic theory we know that

$$v_j \to V, \ C^2_{loc}(R^n)$$

where V is the standard bubble.

With the help of Harnack inequality (see (17) in next section), we can show that any blow up point is isolated blow up point, which is also a critical point of K, and furthermore, using Schoen's trick and $n - 2$-flatness condition for K, the isolated blow up point is in fact a simple blow up point. Then with the help of Pohozaev formula, we show that any positive solution to (3) is uniformly bounded (see [32] or [19]).

By now the argument for uniform bound of solutions is standard so we only sketch its proof. Assume that Theorem 2 is not true. Then using the result in previous section, we know that there is a bounded, convex smooth domain $\Omega \subset D_+$ such that Theorem 2 is true outside Ω. Therefore, there exists a sequence of solutions u_j such that

$$\max_{\Omega} u_j \to \infty, \ as \ \infty.$$

Define

$$\mathbf{S} := \{q \in \Omega; \exists x_j \in \Omega \ s.t. \ u_j(x_j) \to \infty \ and \ x_j \to q\},$$

which is the set of blow-up points of $\{u_j\}$. Using Schoen's selection method, we can choose x_j (where $x_j \to q$) as the local maximum point of u_j so that the assumption for the Harnack inequality is true in an uniform ball $B_{3R}(q)$ (otherwise, we can use another blow up sequence of solutions). Using the Harnack inequality in the ball $B_R(q)$ we know that the energy of u_j is uniformly bounded (see page 975 in [19]), hence, $\mathbf{S}$ is a finite set. Denote by

$$\mathbf{S} = \{q_1, ..., q_m\}.$$

We remark that using the Pohozaev identity we know that q_k's are the critical points of the function K. Anyway, we can choose $\sigma < \frac{1}{4}\min_{k \neq l}|q_k - q_l|$ such that u_j is uniformly bounded in the domain $\Omega_\sigma = \Omega_1 - \cup_{k=1}^m B_\sigma(q_k)$ where Ω_1 is any bounded convex bounded domain containing some q_k. Then using the n-2 flatness condition as in the same argument in the proof of Theorem 1.9 in [38], we can complete the proof.

This shows that u is uniformly bounded in Ω. Hence u is uniformly bounded in the whole space R^n.

5. HARNACK INEQUALITY

We shall outline the blow up argument and the Pohozaev formula to get the uniform bound for solutions to (3).

Let $\Omega \subset R^n$ be a bounded smooth domain. Assume that K is a positive bounded smooth function on Ω. Let $\mu = \mu_j$, $K(x) = K_j(x)$, and $u = u_j$ satisfy

$$(8) \qquad -\Delta u = \mu u + K(x)u^p, \;\; u > 0 \;\; in \;\; \Omega \subset R^n,$$

Let $x_0 = 0$ be a blow up point of $\{u_j\}$. The point 0 is called a simple blow up point if there are a constant c and a sequence of local maximum point x_j of u_j such that

$$(9) \qquad\qquad 0 = \lim_{j \to \infty} x_j,$$

and

$$(10) \qquad u_j(x_j + x) \leq cU_{\lambda_j}(x), \;\; for \; |x| \leq r_0,$$

where $r_0 > 0$ is independent of j, $\lambda_j = u_j(x_j)^{-\frac{2}{n-2}}$ tends to zero as $j \to \infty$ and

$$(11) \qquad\qquad U_\lambda(x) = (\frac{\lambda}{\lambda^2 + |x|^2})^{\frac{n-2}{2}}$$

Note that

$$(12) \qquad\qquad \Delta U_\lambda + n(n-2)U_\lambda^{\frac{n+2}{n-2}} = 0, \;\; in \;\; R^n.$$

It is easy to see that for the *simple blow up* point 0 we have

$$(13) \qquad u_j(x_j + x) \leq cu_j(x_j)^{-1}|x|^{2-n}, \;\; for \; |x| \leq r_0,$$

Using $\lambda^2 + |x|^2 \geq 2|x|\lambda$, we have

$$(14) \qquad\qquad U_\lambda(x) \leq (2|x|)^{\frac{2-n}{2}}.$$

With these observations, we say that $x_0 = 0$ is a *isolated blow up* point of $\{u_j\}$ if

$$(15) \qquad\qquad u_j(x_j + x) \leq c|x|^{\frac{2-n}{2}}, \quad for \ |x| \leq r_0,$$

Using a scaling, we know that the spherical Harnack inequality holds for each $r \in (0, r_0)$. That is, there exists a uniform constant $C > 0$ such that

$$\max_{|x-x_j|=r} u_j(x) \leq C \min_{|x-x_j|=r} u_j(x).$$

We shall show the following crucial estimate

Theorem 8. *Assume that K satisfies the condition as in Main result 1, and there exists a constant $C_2 > 0$ such that*

$$(16) \qquad\qquad \max_{|x|\leq R} u \geq C_2 \max_{|x|\leq 3R} u$$

Then we have the following Harnack inequality that

$$(17) \qquad\qquad \max_{|x|\leq R} u \min_{|x|\leq 2R} u \leq C R^{2-n}$$

for some uniform constant $C > 0$.

The estimate above is important since it implies that any blow up point is isolated blow up point as wanted. It also follows from it that the uniform energy finite property for the solution in the ball B_R. To apply our Harnack inequality (17) near an isolated critical point of K, which is assumed to be a blow up point of a sequence of solutions $\{u_j\}$, we need to verify the assumption (16). Actually, this can be done by move the center of local maximum point of u_j. In fact, assume that $u_j(y_j) \to \infty$ as $y_j \to 0$. We find a ball $B_{8R}(0)$. Take $u_j(z_j) = \max_{\bar{B}_{8R}(0)} u_j(x)$. Then $z_j \to 0$. We define $\bar{u}_j(x) = u_j(z_j + x)$ and then $\bar{u}_j$ satisfies our assumption (16) in the ball B_{2R}. We can apply the Harnack inequality to $\bar{u}_j$.

Assume (17) is not true. Then we have $R_j \to 0$ and solutions u_j corresponding to the data $(\mu_j, K_j(x))$ such that

$$\max_{|x|\leq R_j} u_j \min_{|x|\geq 2R_j} u_j \geq j R_j^{2-n}.$$

Let $y_j : |y_j| \leq R_j$ be such that

$$u_j(y_j) = \max_{|x|\leq R_j} u_j.$$

Assume that $y = \lim y_j$. Let $M_j = u_j(y_j)$ and $\tau_j = M_j^{-\frac{n-2}{2}}$. Then it is easy to see that $M_j \to \infty$. Let

$$v_j(x) = v_{y_j}(x)$$

be the blow up sequence for (u_j). Then we have a subsequence, still denoted by v_j, which is convergent to the standard bubble $U(x)$ in nay large ball in

R^n and for any $\epsilon > 0$, there exists a constant $\eta = \eta(\epsilon)$ such that on an unform size of r it holds

$$\min_{|x| \le r} v_j \le (1 + \epsilon)U(r).$$

(This can be shown by arguing by contradiction again).

Following the moving sphere method used by Li-Zhang [34] (see also Chen -Lin [19]), we can claim that y is the critical point of $K(x) = \lim_j K_j(x)$.

Now we let $\beta_j = |\nabla K_j(y_j)|$. Then we have $\beta_j \to 0$. Again following the moving sphere method used by Li-Zhang, we can show that there is a positive constant $C > 0$ such that

$$\beta_j^{1/(\theta-1)} \le C\tau_j.$$

With all these preparation, we can further show as in Li-Zhang [34] (see also Chen-lin [20]) that the Harnack inequality (17) is true.

It is easy to see that the Harnack inequality (17) implies that there is a uniform constant $C(R) > 0$ such that

$$|\nabla u|_{L^2(B_R)} + |u|_{L^{2n/(n-2)}(B_R)} \le C.$$

The importance of the Harnack estimate is that it implies that the blow up points for u_j are isolated and finite in the ball B_R.

Using the flatness condition for K, we can follow Schoen's localization trick (using the Pohozaev identity) to show that u is uniformly bounded in B_R.

6. ON THE PROOF OF HARNACK INEQUALITY

We shall argue by contradiction. Without loss of generality, we assume that $R = 1$ and set $B = B_1$. Then we have a sequence (u_j) satisfying (3) on B_3 with K and μ replaced by K_j and $\mu_j \in [0, L]$ such that

$$(18) \qquad\qquad \max_{\bar{B}_1} u_j \min_{\bar{B}_2} u_j \ge j.$$

Let y_j be the maximum point of u_j on $\bar{B}_1$. Consider $u(x) = u_j(\frac{1}{2}x + y_j)$ on the ball B in Lemma 17 and $a = (n-2)/2$. Then we find a maximum point x_j of the function

$$u(x)(1 - |x|)^a$$

such that for $\sigma_j = \frac{1}{2} - |x_j - y_j| \le 1/2$,

$$(19) \qquad\qquad u_j(x_j) \ge 2^{-a} \max_{\bar{B}(x_j, \sigma_j/2)} u_j$$

and

$$(20) \qquad\qquad \sigma_j^a u_j(x_j) \ge 2^a u_j(y_j).$$

The last inequality implies that $u_j(x_j) \ge u_j(y_j)$. Let

$$\lambda_j = \frac{1}{2} u_j(x_j)^{1/a} \sigma_j$$

and
$$M_j = u_j(x_j).$$

Then by (18) and (20) we have
$$4\lambda_j \geq u_j(y_j)^a \geq (u_j(y_j) \min_{\bar{B}_2} u_j)^{1/2a} \geq j^{1/2a} \to \infty.$$

We let
$$v_j(y) = M_j^{-1} u_j(x_j + M_j^{-1/a} y), \ \ |y| \leq M_j^{1/a} \to \infty.$$

By direct computation, u_j satisfies that
$$-\Delta u = \mu_j M_j^{-2/a} u + K_j(y) u^p, \ \ \ |y| \leq M_j^{1/a}.$$

Note that $v_j(0) = 1$ and by (19,)
$$\max_{|y| \leq \lambda_j} v_j \leq 2^a v_j(0) = 2^a.$$

Applying the standard elliptic theory, we may assume that
$$v_j \to v; \ \ in \ \ C_{loc}^2(R^n)$$

where v satisfies
$$-\Delta v = \lim_j K_j(x_j) v^p, \ \ in \ \ R^n.$$

Using the classification theorem of Caffarelli et al. [13], we know that v is radially symmetric about some point x_0, which is the only maximum point of v, and $v(y)$ decays like $|y|^{2-n}$ near ∞. With the help of the data of v, we may assume that y_j is the local maximum point of v_j such that
$$x_0 = \lim_j (y_j - x_j) M_j^{1/a}.$$

Then we can re-define v_j at the center y_j. This is the localization blow up trick of R.Schoen.

Again, without loss of generality, we assume that $\lim_j K_j(x_j) = n(n-2)$. Then we have
$$U(y) = (1 + |y|^2)^{-a}.$$

Recall that, for the Kelvin transformation
$$y \to y^\lambda = \lambda^2 y / |y|^2,$$

we have
$$u^\lambda(y) = (\frac{\lambda}{|y|})^{n-2} u(y^\lambda),$$

we have
$$(21) \qquad \Delta u^\lambda(y) = (\frac{\lambda}{|y|})^{n+2} \Delta u(y^\lambda).$$

Note that by direct computation, we have
$$(22) \qquad U(r) - U^\lambda(r) = (1 - \lambda)(1 - \frac{\lambda}{r}) 0(r^{2-n}), \ \ r = |y| \geq \lambda.$$

Note that on one hand, $B(y_j, \frac{1}{2}) \subset B_2$ and

$$-\Delta u_j \geq 0,$$

and by the maximum principle, we have $\min_{\bar{B}_r} v_j$ is monotone non-increasing in r.

On the other hand, by (18), we have

$$\min_{\bar{B}(y_j, \frac{1}{2})} u_j \geq \min_{\bar{B}_2} u_j.$$

Then (18) gives us that

$$\min_{2|y| = M_j^{1/a}} v_j(y)|y|^{n-2} \to \infty.$$

Hence, we can choose $\epsilon_j \to 0$ such that

$$\min_{|y| = \epsilon_j M_j^{1/a}} v_j(y)|y|^{n-2} \to \infty,$$

and

$$T_j := \epsilon_j M_j^{1/a} \to \infty.$$

We now use the Kelvin transformation

$$y \to y^\lambda = \lambda^2 y / |y|^2,$$

where $\lambda \in [.5, 2]$, to the function v_j.

Let

$$v_j^\lambda(y) := \left(\frac{\lambda}{|y|}\right)^{n-2} v_j(y^\lambda).$$

Using the formula (21), we compute that v_j^λ satisfies

$$-\Delta v = \frac{\mu_j M_j^{-2/a} \lambda^4}{|y|^4} v + K_j(x_j + M_j^{-1/a} y^\lambda) v^p, \quad |y| > 0(M_j^{-1/a}).)$$

Let $w_\lambda = v_j - v_j^\lambda$. Then we have

(23)

$$L_\lambda(v_j) w_\lambda := \Delta w_\lambda + \mu_j M_j^{-2/a}\left(1 - \frac{\lambda^4}{|y|^4}\right) w_\lambda + K_j(y_j + M_j^{-1/a} y) P(v_j)^{2/a} w_\lambda = Q_\lambda, \quad in \ \Sigma_\lambda,$$

where $\Sigma_\lambda := B(0, T_j) - \bar{B}_\lambda$,

$$P(v_j)^{2/a} := (v_j^p - (v_j^\lambda)^p) / w_\lambda,$$

and

$$Q_\lambda = -(K_j(x_j + M_j^{-1/a} y) - K_j(x_j + M_j^{-1/a} y^\lambda)) w^p.$$

Note that in Σ_λ, we have

(24) $$Q_\lambda = 0(M_j^{-1/a} r^{-1-n}).$$

This fact will be used later.

We want to use two ways to get a contradiction. One is to show that for any $\epsilon > 0$, there exists a $\delta_0 = \delta_0(\epsilon)$ such that the inequality

$$(25) \qquad \min_{|y| \leq r} v_j(y) \leq (1 + \epsilon)U(r)$$

holds for $0 \leq r \leq \delta_0 T_j$. Once (25) is established, we get a contradiction by our assumption (18). In fact,

$$\min_{|y| \leq T_j} v_j(y) \geq M_j^{-1} \min_{|y| \leq \epsilon_j} u_j(y) \geq \frac{j}{M_j^2 \epsilon_j^{n-2}} \geq j T_j^{2-n},$$

and

$$\geq \frac{j}{\delta_0^{n-2}} U(\delta_0 T_j),$$

which is absurd to (25). We shall see that by using the companion function, we can obtain the following result.

Proposition 9. *For any $\epsilon > 0$, there exists a $\delta_0 = \delta_0(\epsilon)$ such that the inequality*

$$(26) \qquad \min_{|y|=r} v_j(y) \leq (1 + \epsilon)r^{2-n}$$

holds for $r \in (\lambda, \delta_0 M_j^{2/a^2})$.

For dimension $n \geq 5$ we have

Proposition 10. *Assume that $n \geq 5$ and assume that*

$$\max_{\bar{B}} u \geq C \max_{\bar{B}_2} u$$

for some uniform constant $C > 0$. For any $\epsilon > 0$, there exists a $\delta_0 = \delta_0(\epsilon)$ such that the inequality

$$(27) \qquad \min_{|y| \leq r} v_j(y) \leq (1 + \epsilon)U(r)$$

holds for $r \leq \delta_0 M_j^{2/a^2})$.

The case when $n = 4$ will be treated separately. With these preparation, we can proceed the proof by the other one method, which is the following. Note that from (21), we have $U(r) - U^\lambda(r) > 0$ for $r > \lambda$ and $\lambda < 1$, and $U(r) - U^\lambda(r) < 0$ for $r > \lambda$ and $\lambda > 1$. This is a very important fact for us to get a contradiction by moving sphere method. In fact, we shall show that the approximation $v_j - v_j^\lambda > 0$ (of $U(r) - U^\lambda(r)$) on Σ_λ for some $\lambda > 1$ (which is very near to 1) to get a contradiction. Life will be too simple if this can be easily obtained. However, to obtain this, we shall construct a companion function h_λ such that

$$(28) \qquad h_\lambda = 0. \ \ on \ \ \partial B_\lambda$$

and

$$(29) \qquad h_\lambda(x) = \circ(1)r^{2-n}, \ \ on \ \ \Sigma_\lambda.$$

In other word, h_λ is a small perturbation.

Firstly, we consider the boundary condition. By (29) we have that

$$(30) \qquad w_\lambda + h_\lambda > 0, \quad on \ \partial\Sigma_\lambda - \partial B_\lambda$$

We want to show

$$w_\lambda + h_\lambda > 0, \quad on \ \Sigma_\lambda.$$

Hence, to use the maximum principle trick, we need to know that

$$L_\lambda(w_\lambda + h_\lambda) \leq 0, \quad in \ \Sigma_\lambda$$

This requires a subtle construction of h_λ. One remark is that we actually only require

$$L_\lambda(w_\lambda + h_\lambda) \leq 0$$

on the part where $w_\lambda + h_\lambda < 0$. Note that on the region where $v_j > 2v_j^\lambda$, we clearly have $w_\lambda + h_\lambda > 0$. Let

$$\Omega(\lambda) = \{y; v_j(y) \leq 2v_j^\lambda(y)\}.$$

We want to construct h_λ such that

$$(31) \qquad L_\lambda h_\lambda + Q_\lambda \leq 0, \quad in \ \Omega(\lambda).$$

This will be done in the next section.

Secondly, we consider the initial step to use the moving sphere method. From (28-29) and because of $w_{\lambda_0} >> 1$ for $\lambda_0 < 1$ near to 1, we know that

$$(32) \qquad w_{\lambda_0} + h_{\lambda_0} > 0, \quad in \ \Sigma_{\lambda_0}$$

for $\lambda_0 < 1$ but near to 1. So the first step for us to use moving sphere method is done.

Once we start the moving sphere method, we can not stop until (31) can not be held. Anyway, we have h_λ such that (20) is true for all λ near to 1. Then we can use the maximum principle to get that

$$w_{\lambda_1} + h_{\lambda_1} > 0$$

for some $\lambda_1 > 1$, which gives us a contradiction as we wanted. Therefore, (17) is true for general case.

To end this section, let's prove the Harnack inequality (17) in dimension three.

Note that the estimate (27) gives a contradiction to our assumption (18) when $n = 3$. However, it is weaker when $n \geq 4$. As a warm-up, we now prove Proposition 9.

Proof. Assume that (27) is not true. Then there exist a constant $\epsilon_0 > 0$ and a sequence $\delta_j \to 0$ of positive numbers such that for some $r_j \leq \delta_j M_j^2/a^2$,

$$(33) \qquad \min_{|y|=r_j} v_j(y) > (1 + \epsilon_0)r_j^{2-n}.$$

We shall use the moving method in the domain $\Sigma_\lambda = B(0, r_j) - \bar{B}_\lambda$. Since $v_j \to U$ in $C^2_{loc}(R^n)$, we must have $r_j \to \infty$. Let $\alpha \in (2, n)$ and define, for $r = |y| \geq \lambda$,

$$g(r) = g_\alpha(r) = \frac{1}{(2-\alpha)(n-\alpha)}[\lambda^{2-\alpha} - r^{2\alpha}] - \frac{\lambda^{n-\alpha}}{(n-2)(n-\alpha)}[r^{2-n} - \lambda^{2-n}].$$

Note that, for $r \geq \lambda$,

$$g(\lambda) = g'(\lambda) = 0,$$

and

$$-C(n, \alpha) \leq g(r) \leq 0.$$

The function g satisfies

$$\Delta g = -r^{-\alpha}, \quad in \ |y| \geq \lambda.$$

Using (24), for large constant $Q > 0$ and for

$$h = QM_j^{-1/a}g = \circ(1)r^{2-n} < 0, \quad in \ \Sigma_\lambda,$$

and

$$L_\lambda(v_j)(w_\lambda + h) = 0(M_j^{-1/a}r^{-1-n}) + \Delta h + P(v_j)^{2/a}h,$$

which is

$$\leq 0(M_j^{-1/a}r^{-1-n}) + \Delta h < 0, \quad in \ \Sigma_\lambda.$$

Using the fact that for $|y| = r_j$ and for $\lambda \in [1 - \epsilon_1, 1 + \epsilon_1]$ with some small ϵ_1, $w_\lambda + h > 0$, we can use the maximum principle to get that

$$w_\lambda + h > 0, \quad in \ \Sigma_\lambda.$$

This is contradiction to the fact that

$$w_\lambda + h \approx v_j - v^\lambda \to U(r) - U^\lambda(r) < 0, \quad for \ r > \lambda > 1$$

in $C^2_{loc}(R^n)$. Hence (27) is true. $\qquad\qquad\qquad\qquad \square$

As we pointed out before, in dimension three, Proposition 9 implies the Harnack estimate, which was firstly obtained by Schoen when $\mu = 0$. The argument above is a good lesson for other dimensions. Anyway, we have

Theorem 11. *For $n = 3$ and any positive $C^1(B_3)$ function K, the Harnack inequality (17) is true for solutions to (3).*

We should say here that with suitable assumption on the set of critical points of K, we can use Pohozaev identity trick to show the uniform bound on the solution set to (3). Since this case is routine, we just state the result below.

Theorem 12. *For $n = 3$, $\mu \in [0, L]$, and any positive $C^2(B_3)$ function K with $0 < C_1 \leq K \leq C_2 < \infty$, $|K|_{C^1(B_2)} \leq C_2$, and at the critical point $x \in B_2$, $\Delta K(x) \neq 0$. Then there is a uniform constant C such that for any solution to (3) on B_3, we have*

$$|u|_{C^2(B)} \leq C.$$

i.e., the uniform bound on the solution set to (3).

7. The construction of h_λ

Observe that in Σ_λ, we have the expansion

$$Q_\lambda = \sum_{i=1}^{3} M_j^{-ai} \sum_{|\alpha|=i} \frac{1}{\alpha!} K((\alpha)(x_j) y^\alpha (\frac{\lambda^{2i}}{|y|^{2i}} - 1)(v_j^\lambda(y)^p) + 0(M_j^{4a}|y|^{2-n}),$$

where we have used the estimate $v_j^\lambda(y) = 0(|y|^{2-n})$. Note that suing the convergence property

$$v_j \to U, \ \ C_{loc}^2(R^n),$$

and so v_j^λ is close to U^λ, we can write it as

$$Q_\lambda = \sum_{i=1}^{3} M_j^{-ai} \sum_{|\alpha|=i} \frac{1}{\alpha!} K((\alpha)(x_j) y^\alpha (\frac{\lambda^{2i}}{|y|^{2i}} - 1)(U^\lambda(y)^p) + 0(M_j^{4a}|y|^{2-n}), \ \ in \ \Sigma_\lambda.$$

Again, we shall write the above expansion as

$$Q_\lambda = Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + 0(M_j^{4a}|y|^{2-n})$$

with

$$Q_1 = M_j^{-1/a}((\frac{\lambda}{r})^2 - 1)(U^\lambda)^p \sum_k \theta_k,$$

(34)

$$Q_2 = M_j^{-2/a} r^2 ((\frac{\lambda}{r})^4 - 1)(U^\lambda)^p (\sum_{kl\!\!/} \partial_{kl} K_j(y_j)\theta_k\theta_l +$$

$$+ \frac{1}{2} \sum_k \partial_{kk} K_j(x_j)(\theta_k^2 - 1/n)),$$

$$Q_3 = M_j^{-3/a} r^3 ((\frac{\lambda}{r})^6 - 1)(U^\lambda)^p (\frac{1}{6} \sum_k \partial_k^3 K_j(y_j)(\theta_k^3 - \frac{3}{n+2}\theta_k) +$$

$$+ \frac{1}{2(n+2)} \sum_k \partial_k^3 K_j(y_j)\theta_k + \frac{1}{2} \sum_k \partial_{kkl}^3 K_j(y_j)(\theta_k^2\theta_l - \frac{1}{n+2}\theta_l)$$

$$+ \frac{1}{2(n+2)} \sum_{kl\!\!/} \partial_{kkl} K_j(y_j)\theta_l + \sum_{kl\neq m} \partial_{klm} K_j(y_j)\theta_k\theta_l\theta_m),$$

$$Q_4 = M_j^{-2/a} \Delta K_j(x_j) r^2 ((\frac{\lambda}{r})^4 - 1)(U^\lambda)^p,$$

(35)

$$Q_5 = \sum_{i=1}^{3} Q_{5,i} = \sum_{i=1}^{3} M_j^{-ai} \sum_{|\alpha|=i} \frac{1}{\alpha!} K((\alpha)\theta_\alpha r^i (\frac{\lambda^{2i}}{r^{2i}} - 1)((v_j^\lambda)^p - U^\lambda(y)^p),$$

where θ_k $(k = 1, ..., n)$ are the first eigenfunctions of Laplacian operator Δ_θ on the sphere S^{n-1} corresponding to the eigenvalue $n - 1$. Note that by our assumption on K, we have $Q_4 \leq 0$. This interesting property is firstly observed by L.Zhang [53]. Note also that the linear operator $L_\lambda(U)$ satisfies

the assumption of Proposition 19. We now define, according to Proposition 19 for $i = 1, 2, 3$, the functions h_i satisfying

$$L_\lambda(U)h_i = Q_i, \ \ in \ \Sigma_\lambda$$

with the boundary condition and the behavior

$$(36) \qquad\qquad h_i|_{\partial B_\lambda} = 0; \ \ |h_i(y)| = \circ(1)r^{2-n}, \ \ in \ \Sigma_\lambda.$$

Let's now give more precise description about h_i. By using Proposition 20 in appendix B and the fact that

$$-\Delta_\theta \theta_k = (n-1)\theta_k,$$

we have

$$(37) \qquad\qquad h_1 = M_j^{-1/a} \sum_k \partial_k K_j(y_j) \theta_k f_1(r).$$

From the estimate for $_1$ in Proposition 20, we have

$$(38) \qquad |h_1(y)| \leq C_0 |\nabla K_j(y_j)| M_j^{-1/a} r^{2-n}(1 - \frac{\lambda}{r}), \ \ \lambda < r < T_j.$$

Clearly h_1 satisfies (36).

Similarly we have

$$h_2(y) = M_j^{-2/a} f_2(r) (\sum_{kl\!/} \partial_{kl} K_j(y_j) \theta_k \theta_l + \frac{1}{2} \sum_k \partial_{kk} K_j(x_j)(\theta_k^2 - 1/n))$$

and

$$h_3(y) = M_j^{-3/a} [f_3(r)(\frac{1}{6} \sum_k \partial_k^3 K_j(y_j)(\theta_k^3 - \frac{3}{n+2}\theta_k)$$

$$+ \frac{1}{2} \sum_k \partial_{kkl}^3 K_j(y_j)(\theta_k^2 \theta_l - \frac{1}{n+2}\theta_l) + \sum_{k \neq l \neq m} \partial_{klm} K_j(y_j) \theta_k \theta_l \theta_m))$$

$$+ f_4(r)(\frac{1}{2(n+2)} \sum_k \partial_k^3 K_j(y_j) \theta_k + \frac{1}{2(n+2)} \sum_{kl\!/} \partial_{kkl} K_j(y_j) \theta_l)].$$

Using the estimates for f_2, f_3, f_4 in Proposition 20, we have

$$(39) \qquad |h_2(y)| \leq C_0 |\nabla K_j(y_j)| M_j^{-2/a} r^{2-n}(1 - \frac{\lambda}{r}), \ \ \lambda < r < T_j$$

and

$$(40) \qquad |h_3(y)| \leq C_0 |\nabla K_j(y_j)| M_j^{-3/a} r^{3-n}(1 - \frac{\lambda}{r}), \ \ \lambda < r < T_j.$$

Hence, h_2 and h_3 satisfies (36) as wanted.

Let

$$w_3 = w_\lambda - h_1 - h_2 - h_3.$$

Then we have, in $\Omega(\lambda)$,

$$L_\lambda(v_j)w_3 = Q_4 + Q_5 + \sum_{i=1}^{3} Q_{6,i} + 0(M_j^{4a}|y|^{2-n})$$

where

(41) $$Q_{6,i} = K_j((y_j))P(v_j^\lambda)^{2/a} - K_j(y_j + M_j^{-1/a}y)P(v_j)^{2/a}h_i.$$

By this, it is now quite clear that $L_\lambda(v_j)w_3$ has very small positive part. To control this small positive part, we need to construct a non-positive function e_λ such that Δe_λ can control it, which will done in next section. Then, using e_λ non-positive, we have

$$L_\lambda(v_j)e_\lambda \le \Delta e_\lambda$$

which implies that

$$L_\lambda(v_j)(w_3 + e_\lambda) \le L_\lambda(v_j)w_3 + \Delta e_\lambda \le 0.$$

Hence, we can use the maximum principle to $w_3 + e_\lambda$ and then the moving plane method gives us the contradiction wanted provided h_i. $i = 1, 2, 3$ can be neglected, which will be the purpose below and will be studied case by case.

We now follow the argument in Lemma 3.2 of [20] to show

Proposition 13. *There exist $\delta > 0$ and $C > 0$ independent of i such that*

$$v_j(y) \le CU(y), \quad for \quad |y| \le \delta M_j^{1/(2a^2)} := \delta N_j.$$

Proof. Let $G_j(y, \eta)$ be the Green function of the Laplacian operator in the ball $B_j = \{\eta; |\eta| \le N_j\}$ with zero boundary value. For any $\epsilon > 0$, let $\delta_1 > 0$ be chosen as in Proposition 9. Then for $\bar\delta << \delta_1$ small enough (independent of j) we have

$$G_j(y, \eta) \ge \frac{1-\epsilon}{(n-2)|S^{n-1}|}|y - \eta|^{2-n}$$

for $|y| = \delta_1 N_j$ and $\eta| \le \bar\delta N_j$.

Take z_j such that $|z_j| = \delta_1 N_j$ and

$$v_j(z_j) = \min_{|y| \le \delta N_j} v_j(y).$$

Then by Proposition 9, we have

$$(1 + \epsilon)(\delta_1 N_j)^{2-n} \ge v_j(z_j) \ge \int_{B_j} G_j(z_j, \eta)(\mu v_j + K_j v_j^p)d\eta$$

which is bigger than

$$\frac{C(1 - 2\epsilon)}{((\delta_1 + \bar\delta)N_j)^{n-2}} \int_{|\eta| \le \bar\delta N_j} v_j^p d\eta,$$

where C is a dimension constant. Then we have

$$\int_{|\eta| \le \bar\delta N_j} v_j^p d\eta \le C(1 + 4\epsilon).$$

Using $v_j \to U$ in $C_{loc}^2(R^n)$, we may choose $R > 0$ large such that

$$\int_{R \le |\eta| \le \bar\delta N_j} v_j^p d\eta \le C\epsilon.$$

Since $v_j(y) \le 2$, we have

$$\int_{R \le |\eta| \le \bar{\delta} N_j} v_j^{(p-1)n/2} d\eta \le C\epsilon.$$

Hence we use the standard elliptic Harnack inequality ([29] and [31]) to get a uniform constant c such that

$$\max_{|y|=r} v_j \le C \min_{|y|=r} v_j$$

for $r \in [2R, \bar{\delta} N_j/2]$. Using Proposition 9, we have

(42) $$v_j(y) \le CU(y)$$

for $2R \le |y| \le \bar{\delta} N_j/2$. Note that (42) is clearly true for $|y| \le 2R$. Hence we complete the proof of Proposition 13. $\qquad \square$

We want to compare v_j and U. Let $w_j = v_j - U$. Then we have the equation

(43)
$$\Delta w_j + \mu_j M_j^{-2/a} w_j + K_j(y_j + M_j^{-1/a} y) P(v_j)^{2/a} w_\lambda = -\mu_j M_j^{-2/a} U + (n(n-2) - K_j(M_j^{-1/a} y + y_j)) U^p$$

where

$$P_j^{2/a} := (v_j^p - U^p)/w_j.$$

Using the Pohozaev identity (see Proposition 18 in appendix A) and the argument of Lemma 3.3 in [20], we have, for some $\delta_1 \le \delta$,

(44) $$\max_y |v_j(y) - U(y)| \le CM_j^{-1/a}, \quad |y| \le \delta_1 M_j^{-1/(2a^2)}$$

Then we use the standard elliptic estimates to find

(45) $$\sigma_j := |v_j(y) - U(y)|_{C^2(B_3)} \le CM_j^{-1/a}$$

7.1. **Completion of the proof of Theorem 8 when $n = 4, 5$ and some remarks.** In this subsection we prove Theorem 8 when $n = 4$. Since some estimates here will be used in higher dimension, we allow $n \ge 4$ until the end of the proof.

Proof. (of Theorem 8 when $n = 4$)
Write

$$Q_\lambda = Q_1 + 0(M_j^{-1/a^2} r^{-n})$$

and define

$$W_1 = w_\lambda - h_1.$$

Then we have

$$L(v_j)W_1 = 0(M_j^{-1/a^2} r^{-n}) + Q_{5,1} + Q_{6,1}.$$

Since

$$v_j(0) = U(0) = 1, \quad \nabla v_j(0) = \nabla U(0) = 0,$$

we have

$$|v_j(y) - U(y)| \le C\sigma_j |y|^2, \quad in \ B_3.$$

Hence we have

$$|v_j^\lambda(y) - U\lambda(y)| \leq C\sigma_j|y|^{-n}, \;\; for \;\; |y| > \lambda.$$

Using the mean value theorem in Calculus we have

$$v_j^\lambda(y)^p - U\lambda(y)^p = 0(\sigma_j|y|^{-4-n})$$

which is

$$0(M_j^{-1/a}|y|^{-4-n}), \;\; |y| > \lambda.$$

Using the expression of $Q_{5,1}$, we have
(46)
$$|Q_{5,1}| \leq C|\nabla K_j(y_j)|M_j^{-1/a}\sigma_j|y|^{-3-n} \leq M_j^{-1/a}\sigma_j|y|^{-3-n} \leq M_j^{-2/a}|y|^{-3-n}, \;\; |y| > \lambda.$$

Similarly, using the estimate for h_1, we have

$$|Q_{6,1}| \leq CM_j^{-2/a}r^{-4}, \;\; in \;\; \Omega(\lambda).$$

Hence we have

$$L(v_j)W_1 \leq CM_j^{-2/a}r^{-4}, \;\; in \;\; \Omega(\lambda).$$

Introduce $\bar{h} = QM_j^{-2/a}f_3(r)$ for large $Q > 0$. Then we have

$$L(v_j)(W_1 + \bar{h}) \leq 0, \;\; in \; \Omega(\lambda).$$

Note that

$$\bar{h} = \circ(1)r^{2-n} \;\; in \;\; \Sigma_\lambda.$$

So, we can use moving plane method to move to some $\lambda > 1$ with

$$W_1 + \bar{h} > 0, \;\; in \; \Omega(\lambda)$$

which gives us a contradiction when $n = 4$. Hence, when $n = 4$, Theorem 8 has been proved.

$\square$

We now give some important remarks. From the proof above, we have actually proved the following

Proposition 14. *Assume $n \geq 5$. For $\epsilon > 0$, there is a $\delta(\epsilon)$ such that for all $r \leq \delta(\epsilon)M_j^{1/a^2}$,*

$$\min_{|y|=r} v_j(y) \leq (1+\epsilon)U(y).$$

In fact, assume not. Then there exist $\epsilon_0 > 0$, a sequence $\delta_j \to 0$, and a sequence $r_j \leq \delta_j M_j^{1/a^2}$ such that

$$\min_{|y|=r_j} v_j(y) > (1+\epsilon_0)r_j^{2-n}.$$

Let

$$\Sigma_\lambda = B(0, r_j) - \bar{B}_\lambda.$$

Then as in the argument above we have

$$L(v_j)(W_1 + \bar{h}) \leq 0, \;\; in \; \Omega(\lambda).$$

Again, using the moving plane method to get the contradiction.

So Proposition 14 has been proved. Using the same argument as in Lemma 3.2 in [20] with our assumption (16) and the standard elliptic estimate [29], we have

Proposition 15. *Assume $n \geq 5$. For some constant $\delta_2 > 0$ such that*

$$v_j(y) \leq CU(y), \quad |y| \leq \delta_2 M_j^{1/a^2}$$

and

$$|\nabla v_j(y)| \leq C|y|^{1-n}, \quad |y| \leq \delta_2 M_j^{1/a^2}.$$

We let

$$L_j = \frac{1}{2}\delta_2 M_j^{1/a^2}.$$

Assume that $|\nabla K_j(y_j)| \neq 0$ for large some large j. For any $e \in S^{n-1}$, we let

$$\tilde{v}_j(y) = v_j(y + e)$$

and

$$\tilde{K}_j(y) = K_j(M_j^{-1/a}(y + e) + y_j).$$

We choose e a unit vector such that the vector defined by $(\int_{|y| \leq L_j} \nabla_k K_j(y_j) y_k \tilde{v}_j(y)^{p+1})$ (which is non-zero) is lower bounded by $C|\nabla K_j(y_j)|$ for some uniform $C > 0$. We now use the Pohozaev identity (63) in the ball $|y| \leq L_j$ and get

$$\int_{|y| \leq L_j} (\nabla \tilde{K} \cdot y)\tilde{v}_j(y)^{p+1} = 0(M_j^{-2/a}),$$

where the right side consists of boundary terms and lower order terms. Using the definition of $\tilde{K}_j(y)$, we have

$$\int_{|y| \leq L_j} (\nabla \tilde{K}_j \cdot y)\tilde{v}_j(y)^{p+1}$$

$$= M_j^{-1/a} \int_{|y| \leq L_j} (\nabla K_j(M_j^{-1/a}(y + e) + y_j) \cdot y)\tilde{v}_j(y)^{p+1}$$

$$= M_j^{-1/a} \int_{|y| \leq L_j} (\nabla K_j(y_j) \cdot y)\tilde{v}_j(y)^{p+1} + 0(M_j^{-2/a})$$

$$\geq C|\nabla K_j(y_j)|M_j^{-1/a} + +0(M_j^{-2/a}).$$

Hence, we have

(47) $$|\nabla K_j(y_j)| \leq CM_j^{-1/a}.$$

Going back to the equation (43), we see that the right side of (43) is bounded by

$$0(M_j^{-2/a}) + (n(n-2) - K_j(M_j^{-1/a}y + y_j))U^p,$$

and using the second order Taylor's expansion,

$$K_j(M_j^{-1/a}y + y_j)U^p = -\nabla_k K_j(y_j)M_j^{-1/a}y_k U^p + 0(M_j^{-3/a})(|y|^{1-n}).$$

Then the equation (43) is of the form

$$\Delta w_j + \mu_j M_j^{-2/a} w_j + K_j(y_j + M_j^{-1/a}y)P(v_j)^{2/a} w_\lambda = 0(M_j^{-2/a})(1+|y|^{-n}) + 0(M_j^{-3/a})(|y|^{1-n})$$

with the conditions

$$w_j(0) = 0 = |\nabla w_j(0)|.$$

Using this equation and the bound

$$|w_j(y)| \le CU(y), \quad |y| \le L_j,$$

we may follow the argument of Lemma 3.3 in [20] to obtain that

$$(48) \qquad \max_y |v_j(y) - U(y)| \le CM_j^{-2/a}, \quad |y| L_j.$$

Hence, we have

$$(49) \qquad \sigma_j \le CM_j^{-2/a} \quad and \quad |v_j^\lambda(y) - U^\lambda(y)| \le CM_j^{-2/a} r^{-n}.$$

Using this improvement, we show that

Proposition 16. *For $n \ge 6$ and some constant $\delta_4 > 0$, it holds*

$$(50) \qquad v_j(y) \le C(U(y), \quad |y| \le \delta_4 M_j^{-3/(2a^2)}$$

Proof. Note that for $n \ge 5$, we have

$$Q_\lambda = Q_1 + Q_2 + Q_3 + 0(M_j^{-3/a} r^{1-n}).$$

Let

$$W_2 = w_\lambda - h_1 - h_2.$$

Using $Q_4 \le 0$, we obtain that

$$(51) \qquad L(v_j)W_2 \le \sum_{i=1}^{2} Q_{5,i} + \sum_{i=1}^{2} Q_{6,i} + 0(M_j^{-3/a} r^{1-n}).$$

Using (47) and (49) we can improve the bound in (46) into

$$(52) \qquad |Q_{5,1}| \le CM_j^{-8/a}|y|^{-3-n}, \quad |y| > \lambda.$$

Using (49) we bound

$$(53) \qquad |Q_{5,2}| \le CM_j^{-2/a}|y|^2|(v_j^\lambda)^p - (U^\lambda)^p| \le CM_j^{-8/a}|y|^{-2-n}, \quad |y| > \lambda.$$

We now consider the bounds for $Q6,1$ and $Q_{6,2}$. Write

$$K_j(y_j)P(v_j^\lambda)^{2/a} - K_j(M_j^{-1/a}y + y_j)P(v_j)^{-2/a}$$

$$= (K_j(y_j) - K_j(M_j^{-1/a}y + y_j))P(v_j^\lambda)^{2/a} + K_j(M_j^{-1/a}y + y_j)(P(v_j^\lambda)^{2/a} - P(v_j)^{-2/a}).$$

Using (47)

$$(K_j(y_j) - K_j(M_j^{-1/a}y + y_j))P(v_j^\lambda)^{2/a} = 0(M_j^{-2/a}|y|^{-2}) \ \Sigma_\lambda$$

Note that for $r \le L_j$, we have

$$P(v_j)^{-2/a} = P(v_j^\lambda)^{2/a} + 0(M_j^{-2/a})$$

and then

$$K_j(y_j)P(v_j^\lambda)^{2/a} - K_j(M_j^{-1/a}y + y_j)P(v_j)^{-2/a} = 0(M_j^{-2/a}|y|^{n-6})$$

For $r \geq L_j$ and in $\Omega(\lambda)$, we have

$$P(v_j)^{-2/a} = P(v_j^\lambda)^{2/a} + 0(r^{2-n})$$

and then

$$K_j(y_j)P(v_j^\lambda)^{2/a} - K_j(M_j^{-1/a}y + y_j)P(v_j)^{-2/a} = 0(M_j^{-2/a}|y|^{-2}) + 0(r^{-4}),$$

which can be further estimated by

$$|K_j(y_j)P(v_j^\lambda)^{2/a} - K_j(M_j^{-1/a}y + y_j)P(v_j)^{2/a}| \leq 0(r^{-4})$$

where we have used the fact that $r \leq o(1)M_j^{1/a}$. Then using (38) we have, in $\Omega(\lambda)$ with $r > L_j$,

(54) $$|Q_{6,1}| \leq 0(M_j^{-2/a}r^{-2-n})$$

and in $\Omega(\lambda)$ with $r \leq L_j$,

(55) $$|Q_{6,1}| \leq 0(M_j^{-4/a}r^{-4}).$$

Using (39) and (??) we have in $\Omega(\lambda)$ with $r > L_j$,

(56) $$|Q_{6,2}| \leq 0(M_j^{-2/a}r^{-2-n})$$

and in $\Omega(\lambda)$ with $r \leq L_j$,

(57) $$|Q_{6,2}| \leq 0(M_j^{-4/a}r^{-4}).$$

By $0(M_j^{-2/a}r^{-2-n}) = 0(M_j^{-4/a}r^{-4})$ for $r > L_j$, and (51)-(53),(54)-(57) we have, in $\Omega(\lambda)$,

(58) $$L(v_j)W_2 \leq 0(M_j^{-3/a}r^{1-n}) + 0(M_j^{-4/a}r^{-4}).$$

Again we can follow the argument of Lemma 3.3 in [20] to obtain (50) as wanted.   □

Proof. (of Theorem 8 when $n = 5$) The idea is the same as the proof of Theorem 8 when $n = 4$. For large constant $Q > 1$, we let

$$\tilde{h} = QM_j^{-3/a}f_3.$$

Using (58) we have, in $\Omega(\lambda)$,

$$L(v_j)(W_2 + \tilde{h}) \leq 0.$$

Using the moving plane method again we get a contradiction. So the proof of Theorem 8 when $n = 5$ is done.   □

7.2. Completion of the proof of Theorem 8 when $n = 6$.

In this subsection, we assume that $n \geq 6$. Recall that

$$Q_\lambda = Q_1 + Q_2 + Q_3 + Q_4 + \sum_{i=1}^{3} Q_{5,i} + 0(M_j^{4a}|y|^{2-n})$$

and in $\Omega(\lambda)$,

$$L_\lambda(v_j)w_3 = Q_4 + Q_5 + \sum_{i=1}^{3} Q_{6,i} + 0(M_j^{4a}|y|^{2-n})$$

We need to bound $Q_{5,3}$ and $Q_{6,3}$. Using the definition of $Q_{5,3}$ in (35) and the bound (49) we have

$$(59) \qquad |Q_{5,3}| = 0(M_j^{-5/a}r^{-1-n})$$

Similarly, we have by (??) and (??) that for $r < L_j$,

$$(60) \qquad Q_{6,3} = 0(M_j^{-5/a}r^{-3})$$

and for $r > L_j$,

$$(61) \qquad Q_{6,3} = 0(M_j^{-3/a}r^{-1-n}).$$

Note that for $r > L_j$,

$$0(M_j^{-2/a}r^{-2-n}) = 0(M_j^{-4/a}r^{-4}).$$

Using (59)-(61) we obtain in $\Omega(\lambda)$, that

$$L_\lambda(v_j)w_3 = 0(M_j^{-4/a}r^{-4}).$$

For large constant $Q > 1$ and $n = 6$, we let

$$\tilde{h} = QM_j^{-4/a}f_3.$$

Note that for $r \in (\lambda, T_j)$,

$$\tilde{h} = \circ(1).$$

Using the expressions above we have, in $\Omega(\lambda)$,

$$L(v_j)(w_3 + \tilde{h}) \leq 0.$$

Using the moving plane method again we get a contradiction. So the proof of Theorem 8 when $n = 6$ is done.

7.3. Completion of the proof of Theorem 8 when $n \geq 7$. Assume that $n \geq 7$ in this subsection. By assumption on K, we have

$$Q_4 \leq \frac{\Gamma}{2n} M_j^{-2/a} r^2 (\frac{\lambda^4}{r^4} - 1)(U^\lambda)^p.$$

Then by using (59)-(61), we obtain that in $\Omega(\lambda)$ with $r > L_j$,

$$L_\lambda(v_j)w_3 \leq \frac{\Gamma}{2n} M_j^{-2/a} r^2 (\frac{\lambda^4}{r^4} - 1)(U^\lambda)^p + 0(M_j^{-4/a} r^{2-n}) + 0(M_j^{-2/a} r^{-2-n}),$$

and in $\Omega(\lambda)$ with $r \leq L_j$,

$$L_\lambda(v_j)w_3 \leq \frac{\Gamma}{2n} M_j^{-2/a} r^2 (\frac{\lambda^4}{r^4} - 1)(U^\lambda)^p + 0(M_j^{-4/a} r^{2-n}) + 0(M_j^{-4/a} r^{-4}),$$

Hence, since $r \leq \epsilon_j M_j^{1/a}$, we have

$$L_\lambda(v_j)w_3 < 0$$

for $r > \frac{3}{2}\lambda$. Note that in $\Omega(\lambda)$ with $\lambda < r < frac32\lambda$, we have

$$L_\lambda(v_j)w_3 \leq 0(M_j^{-4/a} r^{-4}),$$

whose positive part need to be controlled. To do this, we let f_λ satisfy

$$-\Delta f = 1, \quad in \ \ B_{2\lambda} - B_\lambda$$

with the boundary condition that $f = 0$ on $\partial B_\lambda \bigcup \partial B_{2\lambda}$. Note that there is a constant $C > 0$ such that

$$|f_\lambda| \leq C(1 - \frac{\lambda}{r}), \quad in \ \ B_{2\lambda} - B_\lambda.$$

We extend f_λ smoothly so that $f_\lambda = 0$ outside $B_{3\lambda}$.

For large constant $Q > 1$, we let

$$\tilde{h} = Q M_j^{-4/a} f_\lambda.$$

Then we have

$$-\Delta \tilde{h} = Q M_j^{-4/a}, \quad in \ \ B_{2\lambda} - B_\lambda.$$

We now choose $Q >> 1$ such that

$$\Delta \tilde{h} + 0(M_j^{-4/a} r^{-7/2} \leq 0, \quad in \ \ B_{2\lambda} - B_\lambda.$$

Since

$$|\tilde{h}| \leq C M_j^{-4/a} (1 - \frac{\lambda}{r}), \quad in \ \ B_{2\lambda} - B_\lambda,$$

we have, in $\Omega(\Omega)$,

$$|K_j(M_j^{-1/a} y + y_j) P(v_j)^{2/a} \tilde{h}| \leq \frac{\Gamma}{32n} M_j^{-2/a} r^2 (\frac{1 - \lambda^4}{r^4})(U^\lambda)^p,$$

which leads us to

$$L(v_j)(w_3 + \tilde{h}) \leq 0, \quad \Omega(\lambda).$$

Using the moving plane method as before we get a contradiction. So the proof of Theorem 8 when $n \geq 7$ is done.

We remark that our construction above is similar to that of Li-Zhang [34](see also [53]). Since the appearance of the extra term μu, we need to check out all the detail worked out here.

8. APPENDIX A

Let $B = B_1(0)$ be the unit ball.

Lemma 17. *Let $0 < u \in C^0(\bar{B})$. Let $0 < \phi(r) < 1$ be a function in $C^0[0,1]$ be a decreasing monotone function with $\phi(1)$. Then there exists a point $x \in B$ such that*

$$u(x) \geq \frac{\phi(\sigma)}{\phi(1-2\sigma)} \max_{B_\sigma(x)} u, \quad u(x) \geq \frac{\phi(0)}{\phi(1-2\sigma)} u(0),$$

where $\sigma = (1-|x|)/2 < 1/2$. In particular, for our equation (3) we take $\phi(r) = (1-r)^a$ for $a = \frac{2}{n-2}$, we have

$$2^a u(x) \geq \max_{B_\sigma(x)} u, \quad (2\sigma)^2 u(x) \geq u(0),$$

Proof. Let $v(y) = \phi(|y|)u(y)$. Since $v > 0$ in B and $v = 0$ on ∂B, we have $x \in B$ such that $v(x) = \max_B v > 0$. Note that we have $\sigma = (1-|x|)/2$ and $\sigma \leq 1/2$. Then we have

$$v(x) = \phi(1-2\sigma)u(x) \geq \max_{B_\sigma(x)} v$$

and

$$\geq \phi(\sigma) \max_{B_\sigma(x)} u \geq \phi(1-\sigma) \max_{B_\sigma(x)} u.$$

Similarly, we have

$$\phi(1-2\sigma)u(x) \geq v(0) = \phi(0)u(0).$$

For the special case when $\phi(r) = (1-r)^a$, we have $\phi(0) = 1$, $\phi(1-\sigma) = \sigma^a$ and $\phi(|x|) = \phi(1-2\sigma) = (2\sigma)^a$. $\square$

Let's recall the important Pohozaev formulae for the solutions to (3) in the ball B_R.

Proposition 18. *Let $\nu = x/R$ and*

(62) $$B(R, x, u, \nabla u) = -\frac{n-2}{2} u \partial_\nu u - \frac{R}{2}|\nabla u|^2 + R|\partial_\nu u|^2.$$

Then we have the first Pohozaev identity:
(63)
$$\int_{\partial B_R} B(R, x, u, \nabla u) = \mu \int_{B_R} u^2 - \frac{\mu}{2} \int_{\partial B_R} Ru^2 + \frac{2n}{n-2} \int_{B_R} (x \cdot \nabla K)u^{p+1} - \frac{R}{p+1} \int_{\partial B_R} Ku^{p+1}.$$

We also have the second Pohozaev identity

(64) $$\int_{B_R} \nabla K u^{p+1} = \int_{\partial B_R} [(p+1)(\frac{\lambda}{2}u^2\nu + \nabla u \nabla_{nu} u - \frac{|\nabla u|^2}{2}\nu) + Ku^{p+1}\nu]$$

By now the proof of above Proposition is standard, so we omit its proof.

9. APPENDIX B

Assume that $n \geq 3$. Let $A > 2, b, B, \alpha > 0$ and $\gamma \in [0, n - 2]$ be fixed constants. Assume that the differentiable functions V and H satisfying, for $r \in [1, A]$,

$$(65) \qquad -\alpha^{-1}(1+r)^{-2-\alpha} \leq V(r) \leq n(n+2)U(r)^{2a} + br^{-4}$$

$$(66) \qquad |V'(r)| \leq \alpha^{-1}r^{-3}$$

$$(67) \qquad 0 \leq H(r) \leq Br^{\gamma-n},$$

and

$$(68) \qquad |H'(r)| \leq Br^{\gamma-n-1}.$$

Then, as in [34], we have

Proposition 19. *There exists a unique solution $f = f(r)$ to the problem*

$$f'' + \frac{n-1}{r}f' + (V(r) - \frac{2B}{r^2})f = -H(r), \ 1 < r < A,$$

with the boundary condition

$$f(1) = 0 = f(A).$$

Moreover, for $r \in (1, A)$,

$$0 \leq f(r) \leq Cr^{\gamma+2-n}$$

and

$$|f'(r)| \leq Cr^{\gamma+1-n}$$

where $C > 0$ depends only on n, a, B, γ, and α.

Since the proof is similar to [34], we omit the detail. We use Proposition 19 to prove

Proposition 20. *For each $i = 1, 2, 3$, there exists a unique C^2 radial solution f_i to the problem*

$$\Delta f + (1 - \mu\frac{\lambda^4}{r^4} + K_j(y_j)P(U)^{2/a} - \frac{i(i+n-2)}{r^2})f = r^i(\frac{\lambda^{2i}}{r^{2i}} - 1)(U^\lambda)^p$$

where $\lambda < r < T_j$ with the boundary condition

$$f(\lambda) = 0 = f(T_j).$$

Moreover for $i = 1, 2$, we have

$$(69) \qquad 0 \leq f(r) \leq C_0(1 - \frac{\lambda}{r})r^{2-n}, \ \lambda < r < T_j$$

and for $i = 3$, where C_0 is a dimension constant. Similarly there exists a unique f_4 satisfying

$$\Delta f + (1 - \mu\frac{\lambda^4}{r^4} + K_j(y_j)P(U)^{2/a} - \frac{4(2+n)}{r^2})f = r^3(\frac{\lambda^6}{r^6} - 1)(U^\lambda)^p.$$

where $\lambda < r < T_j$ with the boundary condition

$$f(\lambda) = 0 = f(T_j).$$

For f_4, we have the same bound (69).

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